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Motion in a Bose condensate: VIII. The electron bubble

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Abstract

The Bose condensate model is used to elucidate the motion of the electron bubble in superfluids. An asymptotic expansion is developed for steady subcritical flow. Numerical integration of the coupled nonlinear Schrödinger equations, that describe the evolution of the wavefunctions of the Bose condensate and the impurity, is used to study the nucleation and capture of vortex rings. Because an electron bubble is made oblate by its motion relative to the condensate, the critical velocity for the vortex nucleation is reduced by about 20%, in agreement with experiments.

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(Some figures in this article are in colour only in the electronic version; see `www.iop.org`)

1. Introduction

This paper continues a series of papers devoted to the Bose condensate as applied to superfluid helium; see Roberts and Grant (1971), Grant (1971), Grant and Roberts (1974), Jones and Roberts (1982), Jones *et al* (1986) and Berloff and Roberts (1999, 2000a). These papers will be referred to as I–VII respectively.

The deliberately introduced impurities can be fruitful experimental probes of the structure and dynamics of superfluid helium. Of particular interest is the negative ion which consists of an electron that, through its zero-point motion, carves out a soft bubble of about 16 Å in radius in the surrounding fluid. Careri *et al* (1962) first suggested this form for the negative ion in helium. The induced hydrodynamic mass of such a large structure greatly exceeds its physical mass, and it therefore responds to applied forces as a much more massive ion would. The surrounding helium exerts a net inward pressure across the surface, which is balanced by the zero-point pressure of the electron.

Rayfield and Reif (1964) used an ion time-of-flight spectrometer to determine the dynamics of ion-quantized vortex ring complexes. They observed that above some critical velocity, v_c, the ideal superflow around the ion breaks down, and the moving ions produce vortex rings that then get trapped in their cores. The most extensive set of measurements of the nucleation of vortices by negative ions at elevated pressures came from McClintock group in Lancaster University (see for instance Hendry *et al* 1988).

In spite of such extensive studies the mechanism of vortex nucleation is still not properly understood. The main reason is that there is no truly microscopic picture of superfluid helium available, so the appearance of the vortices 'from nothing', or intrinsic nucleation cannot be derived from first principles. In the absence of such theory the dynamics of the vortices are quite often derived from the Gross–Pitaevskii (GP) model which is assumed to be linked to the condensate fraction of the superfluid. This model has been extensively studied particularly for the motion of ions in a dilute Bose condensate. The condensate is a weakly interacting Bose gas that, in the Hartree approximation, is governed by an equation for the single particle wavefunction $\psi(\boldsymbol{x}, t)$ that was first derived by Ginsburg and Pitaevskii (1958) and Gross (1963); see (1) below. Frisch *et al* (1992) were the first to suggest that the nucleation of vortices takes place when the flow velocity around a moving object exceeds the local speed of sound.

In paper VII, an asymptotic expansion was developed for the flow $\boldsymbol{u}$ round a positive ion moving steadily with a velocity v that is subcritical, i.e., less than the velocity v_c at which the ion begins to nucleate vorticity; the expansion parameter was $\epsilon \equiv a/b \to 0$, where b is the radius of the ion and a is the healing length (defined in section 2). Because v_c is, for the GP model, comparable with the velocity of sound, c, the effects of compressibility cannot be ignored, and it was therefore necessary to generate many terms in this expansion. To observe nucleation, numerical solutions of the condensate equations were derived. These elucidated the underlying processes of boundary layer separation, through which the vortices emerged from the healing layer surrounding the ion.

To study nucleation by the positive ion, it is necessary to model the ion in an *ad hoc* way, by assuming that it is an infinite potential barrier, or by specifying an interaction potential that makes the ion penetrable by the condensate to some degree (Winiecki *et al* 2000). There is no such ambiguity in modelling the negative ion.

In this paper, we generalize paper VII to the more complicated problem of nucleation by a moving electron bubble. By employing a series expansion suitable for $\boldsymbol{u} = \mathcal{O}(c)$, we determine v_c in the limit $\epsilon \to 0$. We show that a moving bubble is oblate, and that this has the effect of decreasing the critical velocity v_c for vortex nucleation, below what it is for a spherical ion. As the velocity of the electron increases, the shedding of vortex rings becomes more and more frequent and irregular. The close proximity of some of the emitted rings allows them to interact with one other and with the bubble. When a ring becomes smaller as a result of this interaction, its velocity increases and it may catch up with the moving bubble. Since, however, the radius of such a ring is less than the radius of the bubble and also because of the axisymmetry of the system, the bubble does not capture the vortex. To observe a capture event, we carried out two 3D numerical integrations. In the first the vortex ring approaches a motionless electron and captures it; in the second a moving ion overtakes a vortex ring of larger radius and is captured by it. In both cases the electron and vortex eventually start moving together as a single 'complex'.

2. The governing equations

In the Hartree approximation, the equations governing the one particle wavefunction of the condensate, ψ, and the wavefunction of the impurity, ϕ, are a pair of coupled equations suggested by Gross (1963) and Clark (1966):

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\psi + (U_0|\phi|^2 + V_0|\psi|^2 - E)\psi \qquad (1)$$

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2\mu}\nabla^2\phi + (U_0|\psi|^2 - E_\mathrm{e})\phi \tag{2}$$

where M and E are the mass and single particle energy for the bosons; μ and E_e are the mass and energy of the electron. The interaction potentials between boson and electron and between bosons are here assumed to be of δ-function form $U_0\delta(\boldsymbol{x}-\boldsymbol{x}')$ and $V_0\delta(\boldsymbol{x}-\boldsymbol{x}')$, respectively. To lowest order, perturbation theory predicts such interaction potentials to be $U_0 = 2\pi l\hbar^2/\mu$ and $V_0 = 4\pi d\hbar^2/M$, where l is the boson-impurity scattering length, and d is the boson diameter. The normalization conditions on the wavefunctions are

$$\int |\psi|^2\,\mathrm{d}V = N \qquad \int |\phi|^2\,\mathrm{d}V = 1 \tag{3}$$

where N is the total number of bosons in the system. The healing length is defined by $a = \hbar(2\rho_s V_0)^{-1/2} = (8\pi\,\mathrm{d}\psi_\infty^2)^{-1/2}$, where $\rho_s = M\psi_s^2 = EM/V_0$ is the mean condensate mass density.

Using the system (1), (2), Grant and Roberts (paper III) studied the motion of a negative ion moving with speed v using an asymptotic expansion in v/c, where c is the speed of sound, so that their leading order flow is incompressible. Treating $\epsilon \equiv (a\mu/lM)^{1/5}$ as a small parameter they calculated the effective (hydrodynamic) radius and effective mass of the electron bubble.

In this paper we shall suppose that the speed v of the electron is comparable with the speed of sound c, so that effects of compressibility are retained. We shall rewrite (1), (2) in the electron reference frame, in which the electron is at rest. We therefore replace E by $E + \frac{1}{2}Mv^2$ in (1) and E_e by $E_\mathrm{e} + \frac{1}{2}\mu v^2$ in (2). Also, in the electron frame, the fluid at infinity moves with velocity v in the negative z-direction and we must therefore require that

$$\psi \to \psi_\infty \exp\left[-\frac{\mathrm{i}Mvz}{\hbar}\right] \qquad \text{for} \quad \boldsymbol{x} \to \infty. \tag{4}$$

The appropriate nondimensional forms of (1) and (2) are therefore

$$\begin{aligned} &\boldsymbol{x} \to \frac{a}{\epsilon}\boldsymbol{x} \qquad && t \to \left(\frac{a^2M}{\hbar\epsilon}\right)t \\ &v \to \left(\frac{\hbar}{aM}\right)U \qquad && \psi \to \psi_\infty\psi \qquad \phi \to \left(\frac{\epsilon^3}{4\pi a^3}\right)^{1/2}\phi. \end{aligned} \tag{5}$$

The nondimensional equations (1) and (2) may be reduced to fluid mechanical form by the Madelung transformation $\psi = R\exp(\mathrm{i}S/\epsilon)$. Equations (1), (2) become

$$\epsilon^2\nabla^2 R - R(\nabla S)^2 = (R^2 + \epsilon^{-2}|\phi|^2 - 1 - U^2) + 2R\partial S/\partial t \tag{6}$$

$$R\nabla^2 S + 2\nabla R\cdot\nabla S + 2\partial R/\partial t = 0 \tag{7}$$

$$\epsilon^2\nabla^2\phi = (q^2R^2 - \epsilon^2k_M^2 - \delta^2U^2)\phi - 2\mathrm{i}\delta\partial\phi/\partial t \tag{8}$$

solutions to which must satisfy

$$\psi \to 1 \qquad \text{for} \quad r \to \infty \qquad \int_V |\phi|^2\,\mathrm{d}V = 4\pi. \tag{9}$$

We do not apply the Madelung transformation to ϕ, because $\delta = \mu/M$ is very small ($\approx 1.4\times10^{-4}$), and can usually be neglected, whereupon, without loss of generality, ϕ may be assumed to be real; we shall obviously discard the δ^2U^2 term in (8). The other nondimensional constants appearing in (6)–(8) are

$$q^2 = \frac{\mu U_0}{MV_0} = \frac{l}{2d} \qquad \epsilon k_M = \left(\frac{\mu E_\mathrm{e}}{ME}\right)^{1/2}. \tag{10}$$

Taking $\rho_\infty = 145.2\ \mathrm{kg\ m^{-3}}$ (Donnelly 1991), we see that $\psi_\infty \approx 1.48$ Å$^{-3/2}$, $E/V_0 \approx 2.18$ Å^{-3}. If $a = 1$ Å, then $V_0 \approx 0.024$ eV Å^3 and $d \approx 1.82$ Å and $E \approx 5.22 \times 10^{-4}$ eV. Assuming $l = 0.6$ Å (Burdick 1965), we find that $\epsilon = 0.187$, $q = 0.41$ and $U_0 \approx 2.87$ eV Å^3. Since $E_\mathrm{e} \approx 0.13$ eV, $k_M \approx 1$.

3. The subcritical solution

The nondimensionalization of (1) and (2) highlights the parameter ϵ, which is fairly small and which enters (6) and (8) as a square. This suggests that (1) and (2) may be solved to good accuracy by an asymptotic solution in which we distinguish between three distinct regimes of the solution: a 'mainstream condensate', a 'mainstream electron', and a 'healing layer' between them. After the solution in the mainstream condensate and the mainstream electron have been found, the solution in a boundary layer can be used to match them. The thickness of this healing layer at the bubble surface is of order $a = \mathrm{O}(\epsilon)$. The success of the asymptotic solution depends less on the smallness of a than on the smallness of a/b, where b is the bubble radius. It is easy to estimate b. In the mainstream bubble, R in (8) can be neglected to leading order, and (since $\delta \ll 1$) this equation reduces to the Helmholtz equation $\nabla^2\phi + k_M^2\phi = 0$, which in spherical coordinates (r, θ, χ) possesses the solution

$$\phi = (k_M^3/\pi)^{1/2}\sin(kr)/kr. \tag{11}$$

This solution is appropriate for a motionless bubble. It shows that the dimensionless radius, b, of that bubble is $b = \pi/k_M \approx \pi$, (or 17 Å in physical units).

Subcritical flow is steady in the ion reference frame, and the Madelung equations are therefore

$$\epsilon^2\nabla^2 R - R(\nabla S)^2 = (R^2 + \epsilon^{-2}\phi^2 - 1 - U^2)R \tag{12}$$

$$R\nabla^2 S + 2\nabla R \cdot \nabla S = 0 \tag{13}$$

$$\epsilon^2\nabla^2\phi = (q^2R^2 - \epsilon^2 k_M^2)\phi. \tag{14}$$

We seek solutions valid for small U and expand the mainstream functions as

$$R(r,\theta) = R_0(r) + \sum_{i=1}^{\infty} U^{2i}\sum_{j=0}^{i} R_{2i,2j}(r)P_{2j}(\cos\theta) \tag{15}$$

$$S(r,\theta) = \sum_{i=1}^{\infty} U^{2i-1}\sum_{j=1}^{i} S_{2i-1,2j-1}(r)P_{2j-1}(\cos\theta) \tag{16}$$

$$\phi = \phi_0(r) + \sum_{i=1}^{\infty} U^{2i}\sum_{j=0}^{i}\phi_{2i,2j}(r)P_{2j}(\cos\theta) \tag{17}$$

where $P_n(x)$ denotes the Legendre polynomial of order n.

The mainstream condensate. In the mainstream

$$\phi_0 = \phi_{2i,2j} = 0. \tag{18}$$

To the leading order in ϵ, the mainstream flow is classical inviscid compressible flow, and is governed by

$$R^2 = 1 + U^2 - (\nabla S)^2 \qquad R^2\nabla^2 S + \nabla R^2 \cdot \nabla S = 0. \tag{19}$$

We substitute the first equation of this system into the second to obtain an equation for S alone. We then expand S in powers of U as in (16) to the U^{11} term to get an estimate for the critical velocity of nucleation, U_c, at which the velocity on the equator exceeds the local speed of sound. The first few terms in (16) become

$$S_{11} = -r - \frac{C_{11}}{r^2} \qquad S_{31} = -\frac{2C_{11}^3}{3r^8} + \frac{8C_{11}^2}{5r^5} + \frac{C_{31}}{r^2} \tag{20}$$

$$S_{33} = -\frac{3C_{11}^3}{11r^8} + \frac{12C_{11}^2}{5r^5} + \frac{6C_{11}}{5r^2} + \frac{C_{33}}{r^4} \tag{21}$$

where C_{ij} are constants of integration.

The mainstream electron. In the mainstream

$$R_0 = R_{2i,2j} = 0. \tag{22}$$

The equations governing ϕ_0 and $\phi_{2i,2j}$ reduce in leading order to Helmholtz equations, to which the physically acceptable solutions are

$$\phi_0 = a_0 j_0(k_M r) \qquad \phi_{2i,2l} = a_{il} j_l(k_M r) \tag{23}$$

where a_0 and a_{il} are constants, $j_\nu(z)$ is the spherical Bessel function $(\pi/2z)^{1/2} J_{\nu+1/2}(z)$, and $J_\nu(z)$ is the Bessel function of the first kind, of order ν and argument z. We define the edge of the impurity bubble to be the locus, $r(\theta)$, of the smallest value of r for that θ at which ϕ vanishes. We determine $r(\theta)$ as an expansion in powers of U

$$r(\theta) = r_0 + \sum_{i=1}^{\infty} U^{2i} \sum_{j=0}^{i} (r_{2i,2j} P_{2j}(\cos\theta)) \tag{24}$$

and we set each term of the Taylor expansion of $\phi(r(\theta), \theta)$ in U equal to zero. This determines the coefficients in (24) in terms of k_M, a_0 and a_{il}. The first few terms in the expansion (24) are

$$\begin{aligned} r(\theta) = \frac{\pi}{k_M}\Bigg(1 + U^2 \frac{3a_{22}}{a_0\pi^2} P_2(\cos\theta) + U^4\Bigg(&\frac{3a_{22}^2(\pi^2-6)}{5a_0^2\pi^4} \\ &+ \frac{-21a_{20}a_{22}\pi^2 + 21a_0a_{42}\pi^2 + 6a_{22}^2(\pi^2-6)}{7a_0^2\pi^4} P_2(\cos\theta) \\ &+ \frac{175a_0a_{44}(21-2\pi^2) + 54a_{22}^2(\pi^2-6)}{35a_0^2\pi^4} P_4(\cos\theta)\Bigg)\Bigg) + \cdots. \end{aligned} \tag{25}$$

This shows that the bubble surface is slightly flattened into a roughly oblate spheroidal form during its motion.

The healing layer; matching the mainstream solutions. The boundary layer equations can only be solved numerically, but we can deduce from them conditions sufficient to match the mainstream solutions. To leading order, in expansions of the form

$$R = R_0 + \epsilon R_1 + \cdots \qquad S = S_0 + \epsilon S_1 + \cdots \qquad \phi = \epsilon\phi_1 + \cdots \tag{26}$$

equation (13) gives

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(R_0^2 \frac{\mathrm{d}S_0}{\mathrm{d}\xi}\right) = 0 \tag{27}$$

where ξ is a scaled coordinate from, and normal to, the bubble surface: $r \approx r(\theta) + \epsilon\xi$. Equation (27) shows that $R_0^2\, \mathrm{d}S_0/\mathrm{d}\xi$ is constant through the boundary layer and, since $R_0 \to 0$ for $\xi \to -\infty$, that constant is zero. Therefore, $S_0 = S_0(\tau)$, where τ is a coordinate tangential to the bubble surface in the plane of symmetry, and increasing with θ, which it replaces because of the non-sphericity of the bubble. The result applies at the next order, so that

$$S_0 = S_0(\tau) \qquad S_1 = S_1(\tau). \tag{28}$$

This justifies applying the condition

$$\frac{\partial S}{\partial \boldsymbol{n}} = 0 \qquad \text{on} \quad r = r(\theta) \tag{29}$$

to the leading order mainstream condensate. Here $\boldsymbol{n}$ is the outward normal to the surface (24) of the electron.

Equations (28) show that, at the leading order in the boundary layer form of (12), $R(\nabla S)^2 = R_0(\partial S_0/\partial\tau)^2$, so that, on omitting the suffices 0 on R_0 and S_0 and the suffix 1, on ϕ_1, we have

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\xi^2} - R\left(\frac{\partial S}{\partial \tau}\right)^2 = (R^2 + \phi^2 - 1 - U^2)R \tag{30}$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}\xi^2} = q^2 R^2 \phi \tag{31}$$

the latter equation being the leading order form of (14).

Condition (29) determines the integration constants $C_{11}, C_{31}, C_{33}, \ldots$ of the mainstream condensate (20), (21) in terms of k_M, a_0 and a_{il} as

$$C_{11} = \frac{\pi^3}{2k_M^3} \qquad C_{31} = -\frac{\pi(20a_0\pi^2 - 27a_{22})}{30a_0k_M^3} \qquad C_{33} = -\frac{27\pi^3(11a_{22} + 4a_0\pi^2)}{110a_0k_M^5}. \tag{32}$$

Multiplying (30) by $\mathrm{d}R/\mathrm{d}\xi$, (31) by $q^{-2}\,\mathrm{d}\phi/\mathrm{d}\xi$ adding and integrating, we obtain

$$\frac{1}{2q^2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}\xi}\right)^2 - \frac{1}{4}R^4 + \frac{1}{2}R^2(1 + U^2 - S'(\tau)^2 - \phi^2) = \text{constant}. \tag{33}$$

Strictly speaking partial derivatives should be used in (33) since ϕ and R depend on both ξ and τ, but the τ dependence is purely parametric, through S. By applying (33) at $\xi = +\infty$, where $R^2 = 1 + U^2 - S'(\tau)^2$ according to (19) and at $\xi = -\infty$, where $R = 0$, we obtain another matching condition between the electron and condensate mainstreams:

$$\frac{1}{2q^2}\left(\frac{\partial\phi}{\partial n}\right)^2 = \frac{1}{4}R^4 \qquad \text{on} \quad r = r(\theta). \tag{34}$$

This condition is tantamount to the condition that the normal stress across the healing layer is continuous; it defines coefficients a_0 and a_{il}:

$$a_0 = \frac{\pi q}{\sqrt{2}k_M} \qquad a_{20} = -\frac{\pi q}{2\sqrt{2}k_M} \qquad a_{22} = -\frac{3\pi^3 q}{2\sqrt{2}(\pi^2 - 3)k_M} \cdots. \tag{35}$$

Finally, we can obtain an expansion of k_M in powers of U from the normalization condition (3) in the form

$$\int_0^{\pi}\int_0^{r(\theta)} \phi^2 r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta = 2 \tag{36}$$

so that

$$k_M^5 = \frac{\pi^3 q^2}{4}\left(1 - U^2 + \frac{2061 + 3\pi^2 - 14\pi^4}{20(\pi^2 - 3)^2}U^4 + \cdots\right). \tag{37}$$

The maximum flow velocity is attained on the equator ($r = r(\pi/2), \theta = \pi/2$) and to leading order is

$$u_{\max} = \frac{f(U)}{g(U)} \tag{38}$$

where

$$f(U) = 1.5U + 1.706\,98U^3 + 8.947\,175U^5 + 56.220\,32U^7 + 390.161U^9 + 2953.94U^{11} + \cdots \tag{39}$$

and

$$g(U) = 1 + 0.327\,5298U^2 + 1.544\,062U^4 + 7.503\,81U^6 + 36.2519U^8 + 181.9392U^{10} + \cdots. \tag{40}$$

The series (39) and (40) appear to converge, though very slowly. We may however regard them as being asymptotic expansions in which the error made in retaining only the terms displayed is less than the last term retained. Van Dyke (1975) has discussed in detail series of this type, and Rica (1999) has employed them for the flow round a circular cylinder in the GP model. If we set

$$u_{\max} = c(\rho) \tag{41}$$

we obtain the critical velocity $U_c \approx 0.34$. Our numerical calculations for $\epsilon = 0.18$ give the critical velocity $U_c \approx 0.32$. (For other definitions of the critical velocity, see the appendix.)

4. Vortex nucleation

In this section we present results from numerical calculations for the axisymmetric, time-dependent flow around the electron and the nucleation of vortex rings from it. We used a different non-dimensionalization of (1), (2):

$$\begin{aligned} &\boldsymbol{x} \to a\boldsymbol{x} \qquad t \to (a^2M/\hbar)t \qquad \phi \to (\epsilon^3/4\pi a^3)^{1/2}\phi \\ &v \to (\hbar/aM)U \qquad \psi \to \psi_\infty\psi. \end{aligned} \tag{42}$$

To keep the ion in the center of the computational box, we transform z to $z - Ut$. Equations (1), (2) become

$$2\mathrm{i}\left(\frac{\partial\psi}{\partial t} - U\frac{\partial\psi}{\partial z}\right) = -\nabla^2\psi + (|\psi|^2 + f|\phi|^2 - 1)\psi \tag{43}$$

$$2\mathrm{i}\delta\left(\frac{\partial\phi}{\partial t} - U\frac{\partial\phi}{\partial z}\right) = -\nabla^2\phi + (q^2|\psi|^2 - k^2)\phi \tag{44}$$

where $k = \epsilon k_M \approx 0.185$ and $f = 1/\epsilon^2$. The normalization condition (9) on ϕ becomes

$$\int |\phi|^2\,\mathrm{d}V = 4\pi/\epsilon^3. \tag{45}$$

To observe and elucidate the process of emission of the vortex rings by the electron we solved (43), (44) by adapting our finite-differences code previously used to solve the GP model of the flow around a moving positive ion (paper VII). On the boundaries for the condensate equation we used the Raymond–Kuo (1984) boundary conditions that allow sound ways to escape. In time stepping, the leap-frog scheme was implemented with a backward Euler step every 100 steps to prevent the even–odd instability. In space we used a fourth order finite difference scheme together with a second order scheme close to the open boundary for both condensate and electron wavefunctions. Because of the smallness of δ, 50 time steps were taken in advancing ϕ before the next time step for ψ was performed. A numerical integration with a larger number of time steps in the impurity equation produced the same result. To minimize the numerical dispersion and to account for a small amount of normal fluid present in superfluid even at very low temperature a small dissipative term was added to (43); see Berloff (1999) for details. The initial condition was chosen as $\psi(\boldsymbol{x}, t = 0) = 0$ for $|\boldsymbol{x} - \boldsymbol{x}_{\rm B}| < b$ but $\psi = \tanh(|\boldsymbol{x} - \boldsymbol{x}_{\rm B}|/\sqrt{2})$ otherwise. The initial condition for the impurity wavefunction was taken as in (11) centered at $\boldsymbol{x}_{\rm B}$. The size of the computation box for axisymmetric problem in cylindrical coordinates (s, z) was $[-50, 50] \times [-100, 100]$ with a computational grid of 400×800 points. For the axisymmetric problem the electron wavefunction equation was integrated in imaginary time (with leap-frog replaced by forward difference) and normalized according to (45) before advancing the condensate equation. The code was tested against the asymptotic solutions found in section 2.

As the velocity, U, increases, but stays subcritical no vortex rings are nucleated. The bubble surface is flattened in accordance with (25). As U becomes supercritical the vortex ring starts to form downstream of the equator on the surface of the bubble. The vortex ring emission follows the same scenario as that observed by Berloff and Roberts (2000a) (paper VII) for the positive ion. After the electron bubble emits a vortex ring, the flow associated with the ring at first makes the mainstream velocity round the ion subcritical everywhere. The self-induced velocity of the ring is less than the velocity of the electron, so that the ring gradually falls astern of the electron and the total fluid velocity builds up until it again reaches criticality on the surface of the impurity; see figure 1.

5. Vortex capture

Rayfield and Reif (1964) observed that, after the moving impurity produced vortex rings, it becomes trapped in one of them. To simulate this process we performed full 3D integrations of (43), (44) for the following configuration: the ion is at rest ($U = 0$) and the vortex ring moves towards the impurity with its own self-induced velocity. The axis of the ring does not coincide with the axis of the impurity. Such a condition is necessary to destroy the axisymmetry of the system. Figure 2 shows the process of capture of the impurity by the vortex ring. Initially, vortex ring of radius 25 and center at $(-25, 5, 0)$ moves with velocity $U \approx 0.09$ (paper IV) towards the stationary negative ion situated at the origin. The Bernoulli effect of the flow created by the moving vortex ring propels the ion and vortex towards one another with a force approximately proportional to s^{-3}, where s is the closest distance between them, similarly to the process of capture of the ion by a straight line vortex (Berloff and Roberts 2000b). As the electron becomes trapped in the vortex core, the flow round the electron bubble acquires circulation that it previously could not possess. When an ion is captured by a straight-line vortex, Kelvin waves are excited that travel along the vortex in each direction away from the ion. In a similar way, we see from figure 2 that waves are set up on a circular ring as it captures the ion.

A similar capture process occurs in another configuration: an ion moving with velocity 0.25 catches up with a vortex ring of radius 30 moving in the same direction with velocity 0.073. The calculations were performed in two stages. First is to prepare the flow round the ion (with a remote vortex ring); this is carried out in the ion reference frame. Since it is moving subcritically, the ion experiences no drag. This solution is used as the initial condition for the second phase, in which the interaction of the ion with the vortex is studied in the laboratory frame, where the fluid at infinity is at rest. The results are shown in figure 3. It can be seen that, as ion approaches the ring, they are attracted to each other and their motions change. The capture process follows in a similar way to that shown for the stationary ion in figure 2.

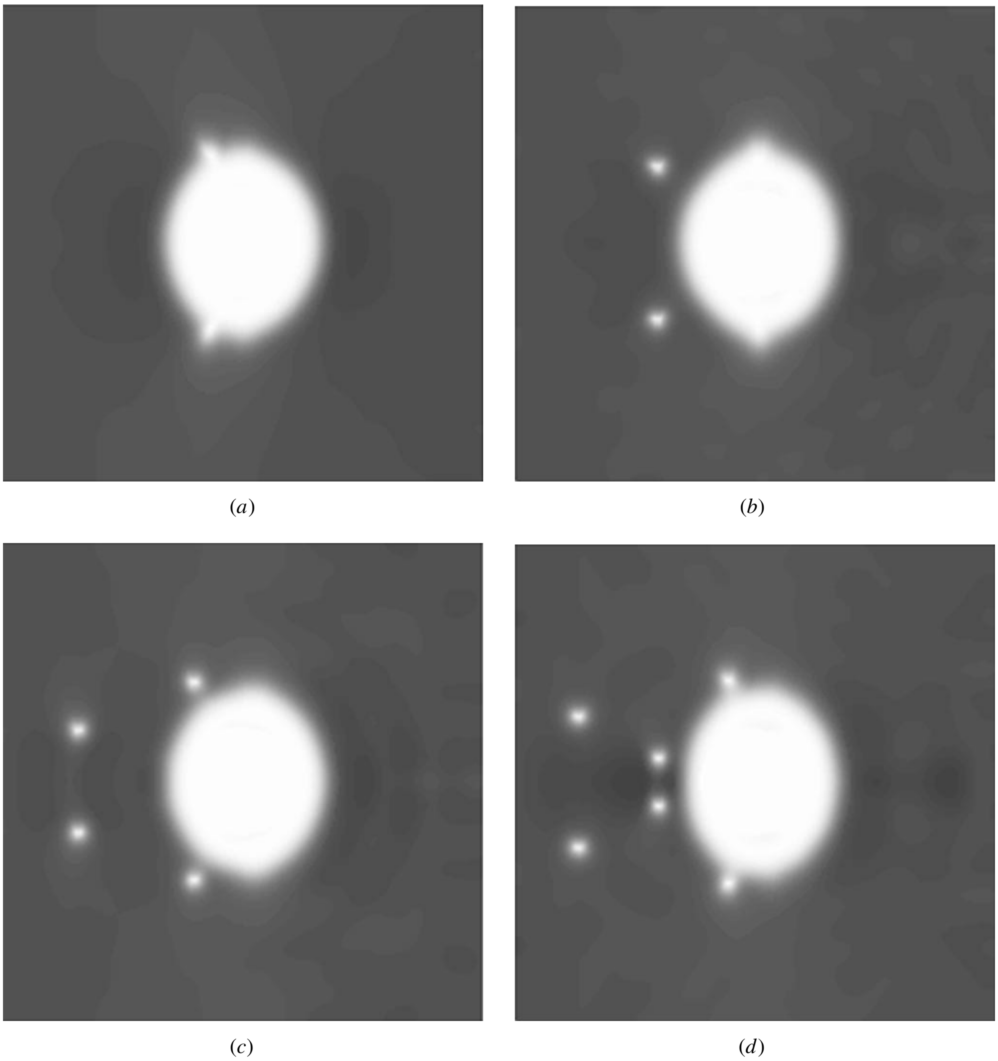

Figure 1. The density plot in a cross-section of the solution of (43), (44) for the flow of the condensate around a negative ion moving to the right with velocity 0.33. The dynamics of the turbulent wake is shown through time snapshots (*a*) $t = 80$, (*b*) $t = 230$, (*c*) $t = 334$, and (*d*) $t = 440$. The initial condition was taken as the subcritical flow with the velocity 0.28.

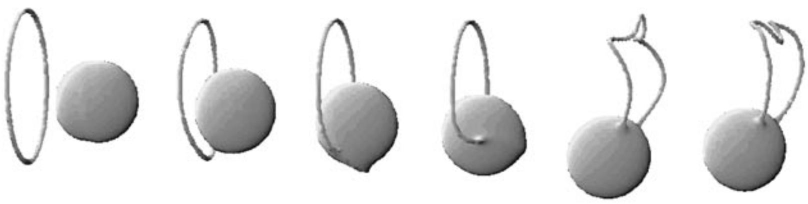

Figure 2. Capture of the stationary ion by the vortex ring: the results of numerical integration of (43), (44) for the isosurface $\rho = 0.2$.

Figure 3. Capture of the moving ion by the vortex ring: the results of numerical integration of (1), (2) for the isosurface $\rho = 0.2$.

Figures 2 and 3 depict only the capture of the ion by a large ring which, we envisage, may have been created by a different ion, moving supercritically in another part of the system. Although it would not have been difficult to do so, we did not study ion capture by smaller rings, a process relevant to situation in which an ion first nucleates a ring and is then swallowed by it. This has been simulated for a positive ion, modeled as a penetrable sphere, by Winiecki and Adams (2000) through the numerical integration of the GP equation, supplemented with the calculation of the drag on the surface of the sphere. In our computations, the ion and condensate are governed by their coupled equations, and no separate calculation of the drag is required.

6. Conclusions

We have studied the motion of the negative ion through the Bose condensate using the Gross–Clark model of the electron bubble. We have established by asymptotic analysis, similarly to the case of the positive ion considered in paper VII, a critical velocity v_c exists; the ion generates vortex rings if its velocity v exceeds v_c.

We have shown that v_c for the negative ion is about 20% less than v_c for the positive ion, in agreement with the experimental findings of Zoll (1976); see also table 8.2 of Donnelly (1991). This reduction may be attributed to the flattening of the electron bubble by its motion through the condensate, as demonstrated in section 3 and by figure 1. The 'equatorial bulge' is created by the difference in pressure between the poles and equator associated with the greater condensate velocity at the latter than at the former. The existence of the bulge also enhances these differences in velocity (and pressure), as compared with a spherical impurity, with the result that, if v is gradually increased from zero, the flow u_e on the equator of the electron bubble attains the velocity of sound before u_e does for the positive ion.

We numerically integrated the Gross–Clark model to demonstrate the nucleation of vortices when $v > v_c$. We also simulated the capture of the negative ion by a vortex ring.

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Appendix. Different criteria for criticality

In paper VII, we proposed that superfluidity is destroyed when

$$u_{\max} = c \tag{46}$$

and if we used this criterion in place of (41) we would find $U_c \approx 0.35$. Conversely, if in paper VII the criterion (41) had been used for the positive ion we would have obtained $U_c \approx 0.38$ instead of the value $U_c \approx 0.41$ quoted in paper VII for $\epsilon = 0$, which was based on (46). The corresponding result for flow past a cylinder (see section 4 of paper VII) would then coincide with that of Rica (1999). It was pointed out to us by W F Vinen (private communication) that the breakdown of superfluidity should also depend on the mass m_{ion}, and that local criteria such as (41) or (46) are tenable only if $m_{\text{ion}} = \infty$. It is not clear (to us) how a criterion dependent on m_{ion} could emerge from anything as simple as the GP or Gross–Clark models. It should however be pointed out that, because in paper VII we specified the velocity of the ion rather than the force on it, we were effectively assuming that $m_{\text{ion}} = \infty$. Moreover, the induced hydrodynamic mass on the negative ion is so large that U_c should not differ significantly from its value for $m_{\text{ion}} = \infty$.

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